SIMULATING MULTIDIMENSIONAL PROBLEMS OF HEAT CONDUCTION WITH PHASE TRANSFORMATIONS

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The special two-dimensional heat conduction problem in the solidification of a melt illustrates how three-dimensional problems with moving boundaries can be simulated on an "SEI-2" static electrical integrator.

The problem is stated as follows: to calculate the transient temperature fields and the progress of the solidification front in a melt–solid system (Fig. 1). It is assumed that phase transformations of the first kind occur here, that the heat transfer by natural convection inside the melt is negligible, and that the heat transfer from the surfaces X = 1 and Y = 1 to the surrounding medium is both radiative and convective.

Mathematically this problem reduces to integrating the following system of differential equations (by virtue of the prevailing symmetry here, only one quadrant of a cross section is considered, Fig. 1):

$$\bar{c}(\Theta)\frac{\partial\Theta}{\partial\tau}=\frac{\partial}{\partial X}\left(\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial X}\right)+\frac{\partial}{\partial Y}\left(\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial Y}\right),\quad \tau>0,\quad 0<X<1,\quad 0<Y<1 \tag{1}$$

with initial and boundary conditions

$$\Theta(0,\ X,\ Y)=f(X,\ Y),\quad \xi(0,\ X,\ Y)=0, \tag{2}$$

$$\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial X}\Big|_{X=0}=\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial Y}\Big|_{Y=0}=0, \tag{3}$$

$$-\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial Y}=\mathrm{Bi}\,(\Theta_{\mathrm{sur}}-\Theta_{\mathrm{amb}})+\frac{\varepsilon\sigma}{\lambda_{\max}}\left[\left(\frac{T_{\mathrm{sur}}}{100}\right)^4-\left(\frac{T_{\mathrm{amb}}}{100}\right)^4\right], \tag{4}$$

$$Y=l_{21},\ 0\leqslant X\leqslant 1,$$

$$-\bar{\lambda}(\Theta)\frac{\partial\Theta}{\partial X}=\mathrm{Bi}\,(\Theta_{\mathrm{пов}}-\Theta_{\mathrm{amb}})+\frac{\varepsilon\sigma}{\lambda_{\max}}\left[\left(\frac{T_{\mathrm{sur}}}{100}\right)^4-\left(\frac{T_{\mathrm{amb}}}{100}\right)^4\right], \tag{5}$$

$$X=1,\ 0\leqslant Y\leqslant 1,$$

$$\left.\begin{aligned}&(\bar{\lambda}(\Theta)\,\mathrm{grad}\,\Theta)|_{\Theta=\Theta_m-0}-(\bar{\lambda}(\Theta)\,\mathrm{grad}\,\Theta)|_{\Theta=\Theta_m+0},\ \mathrm{grad}\,\xi)+L_m\frac{d\xi}{d\tau}=0\\&\Theta(X,\ Y,\ \tau)=\Theta_m\end{aligned}\right\}, \tag{6}$$

$$\xi=\xi(X,\ Y,\ \tau).$$

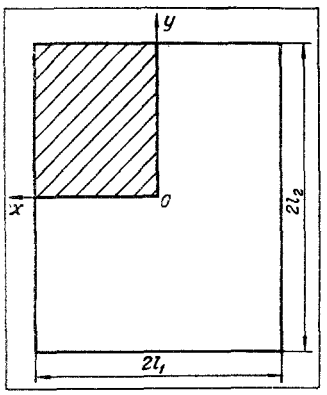

Fig. 1. System of coordinates.

We introduce a new dependent variable

$$\Phi=\int_0^{\Theta}\bar{\lambda}(\Theta)\,d\Theta. \tag{7}$$

Thus, the system of differential equations (1)–(6) can now be rewritten as

$$\frac{\partial\Phi}{\partial\tau}=a(\Theta)\left\{\frac{\partial^2\Phi}{\partial X^2}+\frac{\partial^2\Phi}{\partial Y^2}\right\}, \tag{8}$$

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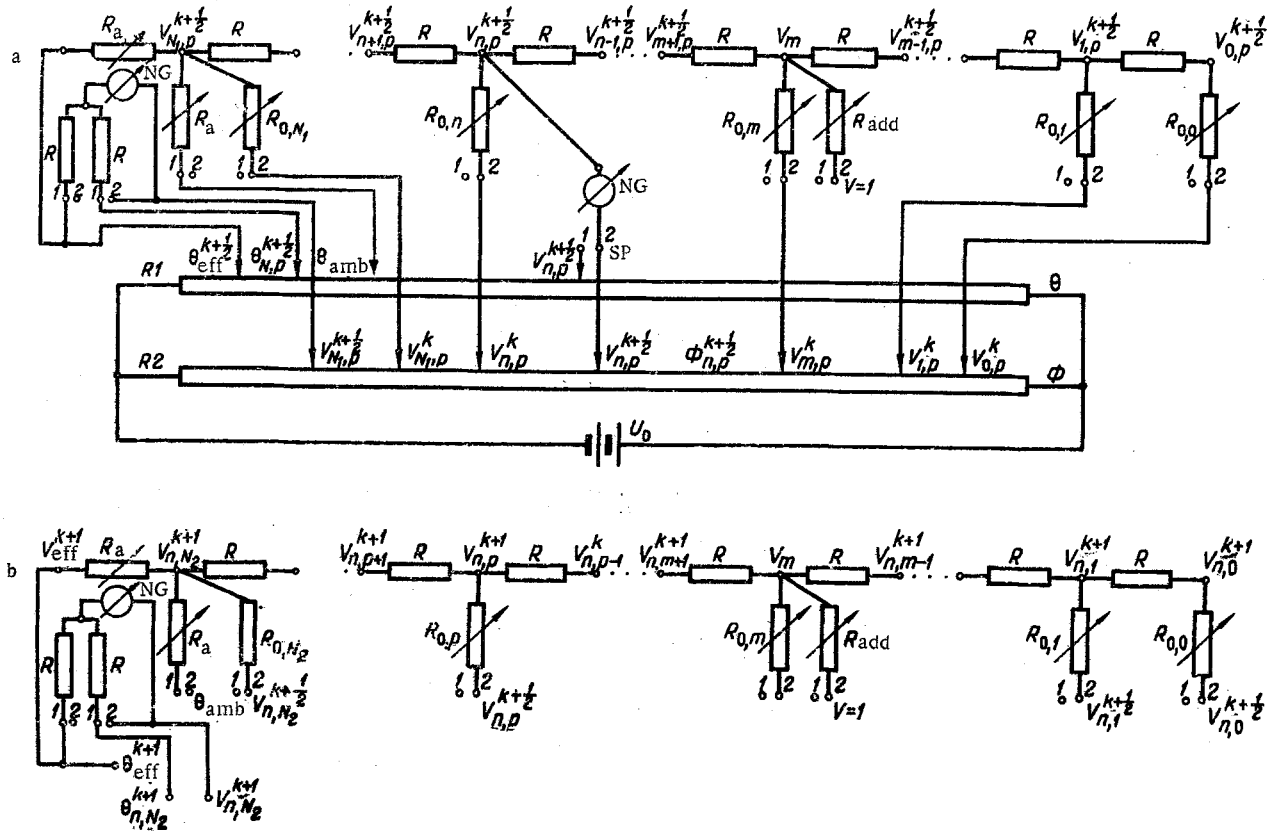

Fig. 2. Schematic diagram of the arithmetic units in the static electrical integrator for calculating the temperature field: on the first fraction step (k + 1/2) (a) and on the second fraction step (k + 1) (b)

$$\tau > 0,\ 0 < X < 1,\ 0 < Y < l_{21},$$

$$\Phi(0,\ X,\ Y) = \varphi(X,\ Y),\ \xi(0,\ X,\ Y) = 0, \tag{9}$$

$$\left.\frac{\partial \Phi}{\partial X}\right|_{X=0} = \left.\frac{\partial \Phi}{\partial Y}\right|_{Y=0} = 0, \tag{10}$$

$$-\frac{\partial \Phi}{\partial Y} = \mathrm{Bi}\,(\Theta_{\mathrm{sur}} - \Theta_{\mathrm{amb}}) + \frac{\varepsilon\sigma}{\lambda_{\max}}\left[\left(\frac{T_{\mathrm{sur}}}{100}\right)^4 - \left(\frac{T_{\mathrm{amb}}}{100}\right)^4\right], \tag{11}$$

$$Y = l_{21},\ 0 \leqslant X \leqslant 1,$$

$$-\frac{\partial \Phi}{\partial X} = \mathrm{Bi}\,(\Theta_{\mathrm{sur}} - \Theta_{\mathrm{amb}}) + \frac{\varepsilon\sigma}{\lambda_{\max}}\left[\left(\frac{T_{\mathrm{sur}}}{100}\right)^4 - \left(\frac{T_{\mathrm{amb}}}{100}\right)^4\right], \tag{12}$$

$$X = 1,\ 0 \leqslant Y \leqslant l_{21},$$

$$\left.\begin{aligned} &\left((\mathrm{grad}\,\Phi)\big|_{\Phi=\Phi_m-0} - (\mathrm{grad}\,\Phi)\big|_{\Phi=\Phi_m+0},\ \mathrm{grad}\,\xi(X,\ Y,\ \tau)\right) + L_m\frac{d\xi}{d\tau} = 0 \\ &\Phi(X,\ Y,\ \tau) = \Phi_m \\ &\xi = \xi(X,\ Y,\ \tau). \end{aligned}\right\} \tag{13}$$

For a numerical solution of system (8)-(13) we use the straight-through computation schedule [1]. As a result, we obtain the following system of finite-difference equations on the first fraction step (k + 1/2):

$$\frac{\Delta X^2}{\Delta\tau}\left(\Phi_{n,p}^{k+\frac{1}{2}} - \Phi_{n,p}^{k}\right) = a\left(\Theta_{n,p}^{k+\frac{1}{2}}\right)\left(\Phi_{n-1,p}^{k+\frac{1}{2}} - 2\Phi_{n,p}^{k+\frac{1}{2}} + \Phi_{n+1,p}^{k+\frac{1}{2}}\right), \tag{14}$$

$$\Phi_{N_1,p}^{k+\frac{1}{2}} - \Phi_{N_1-1,p}^{k+\frac{1}{2}} = \mathrm{Bi}\left(\Theta_{N_1,p}^{k+\frac{1}{2}}\right)\Delta X\left(\Theta_{\mathrm{эф}}^{k+\frac{1}{2}} - \Phi_{N_1,p}^{k+\frac{1}{2}}\right)$$

$$+\frac{1}{2}\frac{\Delta X^2}{\Delta\tau}a\left(\Theta_{N_1,p}^{k+\frac{1}{2}}\right)\left(\Phi_{N_1,p}^{k+\frac{1}{2}} - \Phi_{N_1,p}^{k}\right), \tag{15}$$

$$\Phi_{0,p}^{k+\frac{1}{2}} = \Phi_{1,p}^{k+\frac{1}{2}} + \frac{1}{2} a\left(\Theta_{0,p}^{k+\frac{1}{2}}\right) \frac{\Delta X^2}{\Delta \tau} \left(\Phi_{0,p}^{k+\frac{1}{2}} - \Phi_{0,p}^{k}\right), \tag{16}$$

$$\Phi_{m+1,p}^{k+\frac{1}{2}} - \Phi_m - \left(\Phi_m - \Phi_{m-1,p}^{k+\frac{1}{2}}\right) = L_m \Delta X \left(\frac{\xi^{k+\frac{1}{2}} - \xi^k}{\Delta \tau}\right) \tag{17}$$

and on the second fraction step (k + 1):

$$\left(\Phi_{n,p}^{k+1} - \Phi_{n,p}^{k+\frac{1}{2}}\right) \frac{\Delta Y^2}{\Delta \tau} = a\left(\Theta_{n,p}^{k+1}\right)\left(\Phi_{n,p-1}^{k+1} - 2\Phi_{n,p}^{k+1} + \Phi_{n,p+1}^{k+1}\right), \tag{18}$$

$$\Phi_{n,0}^{k+1} = \Phi_{n,1}^{k+1} + \frac{1}{2} \cdot \frac{\Delta Y^2}{\Delta \tau} a\left(\Theta_{n,0}^{k+1}\right)\left(\Phi_{n,0}^{k+1} - \Phi_{n,0}^{k+\frac{1}{2}}\right), \tag{19}$$

$$\Phi_{n,N_2}^{k+1} - \Phi_{n,N_2-1}^{k+1} = \mathrm{Bi}^*\left(\Theta_{n,N_2}^{k+1}\right)\left(\Theta_{\text{э}\Phi}^{k+1} - \Phi_{n,N_2}^{k+1}\right) + \frac{1}{2} \cdot \frac{\Delta Y^2}{\Delta \tau} a\left(\Theta_{n,N_2}^{k+1}\right)\left(\Phi_{n,N_2}^{k+1} - \Phi_{n,N_2}^{k+\frac{1}{2}}\right), \tag{20}$$

$$\Phi_{n,m+1}^{k+1} - \Phi_m - \left(\Phi_m - \Phi_{n,m-1}^{k+1}\right) = L_m \Delta Y \left(\frac{\xi^{k+1} - \xi^{k+\frac{1}{2}}}{\Delta \tau}\right). \tag{21}$$

The last term in Eqs. (15) and (16) respectively is obtained from the heat balance equation for the $\Delta X/2$ layer adjoining the X = 1 (or X = 0) surface. The resulting system of difference equations is simulated on an "SEI-2" static electrical integrator [2, 3]. The arithmetic unit for the first fraction step is connected into a circuit shown in Fig. 2a with the following values of resistors:

$$R = 1;\ R_{0,n} = \Delta\tau a\left(\Theta_{n,p}^{k+\frac{1}{2}}\right)/\Delta X^2;$$
$$R_a = \frac{1}{\mathrm{Bi}^*\left(\Theta_{N_1,p}^{k+\frac{1}{2}}\right)\Delta X};\quad R_{0,0} = \frac{2\Delta\tau a\left(\Theta_{0,p}^{k+\frac{1}{2}}\right)}{\Delta X^2};\quad R_{0,N_1} = \frac{2\Delta\tau a\left(\Theta_{N_1,p}^{k+\frac{1}{2}}\right)}{\Delta X^2}, \tag{22}$$

and for the second fraction step into a circuit shown in Fig. 2b with the following values of resistors:

$$R = 1;\ R_{0,n} = \Delta\tau a\left(\Theta_{n,p}^{k+1}\right)/\Delta Y^2;\ R_a = \frac{1}{\mathrm{Bi}^*\left(\Theta_{n,N_2}^{k+1}\right)\Delta Y};$$
$$R_{0,0} = \frac{2\Delta\tau a\left(\Theta_{n,0}^{k+1}\right)}{\Delta Y^2};\quad R_{0,N_2} = \frac{2\Delta\tau a\left(\Theta_{n,N_2}^{k+1}\right)}{\Delta Y^2}. \tag{23}$$

The simulation conditions (22)-(23) are based on the correspondence between the equations of voltage-potential distribution at the respective nodes of the arithmetic circuit and the difference equations for the same nodal points in the network field.

The solution is carried out in the following sequence: to the free terminal of each $R_{0,n}(\Theta_{n,p}^{k+(1/2)})$ resistors (Fig. 2a) one applies, through plugs, a voltage potential from the function potentiometer R2*) which is numerically equal to $\Phi_{n,1}$ at the instant of time $\tau = k\Delta\tau$, while to the free terminal of each $R_a(\Theta_{N_1,p}^{k+(1/2)})$ resistor one applies a voltage potential from the linear potentiometer R1 which is numerically equal to Θ_{amb} and $\Theta_{eff}^{k+(1/2)}$. Since the value of $\Theta_{eff}^{k+(1/2)}$ depends on Φ and Θ on the scanned layer of time intervals, it is necessary to correct the value of $\Theta_{eff}^{k+(1/2)}$ and this is achieved by connecting additional resistors R along a null-compensation galvanometer to the (N_1, p) node (Fig. 2a). Iteration is then performed as follows: after the (s + 1)-th iteration for the sought function $\overset{s+1}{\Phi}{}_{n,p}^{k+\frac{1}{2}}$ (or $\overset{s+1}{\Phi}{}_{n,p}^{k+1}$), coefficients $a(\Theta)$ and $\mathrm{Bi}^*(\Theta)$ are calculated from the temperature values $\overset{s}{\Theta}{}_{n,p}^{k+\frac{1}{2}}$ (or $\overset{s}{\Theta}{}_{n,p}^{k+1}$) of the preceeding iteration. It is to be noted that a change from Φ to Θ (or $\Theta \rightarrow \Phi$) is effected on the common switching panel for potentiometers R1 and R2 with an array of plug sockets superposed in space. After the iteration cycle had been

* On the function potentiometer R2 one sets up the relation according to (7), where $\bar{\lambda} = \bar{\lambda}(\Theta)$ may be prescribed in the form of an analytical relation or from a table.

completed (two to three iterations are usually sufficient for convergence within $\pm 0.001\ \Theta_{max}$ precision, $\Theta_{max} = 1$), a voltage-potential distribution numerically equal to $\Phi_{n,1}^{k+(1/2)}$ is established at the nodes of the arithmetic circuit. The solution $\Phi_{n,1}^{k+(1/2)} = V_{n,1}^{k+(1/2)}$ is found on the function potentiometer R2 by the null method, while the sought value of $\Theta_{n,1}^{k+(1/2)}$ is read on the integrator switching panel (the number of the socket holding the search plug SP is numerically equal to $\Theta_{n,1}^{k+(1/2)}$. Thus, the initial temperature distribution and the results of the solution are given in terms of function Θ, although the solution is obtained at the nodes of the arithmetic circuit in terms of function Φ. The temperature distribution on lines $p = 2, 3, \ldots, N_2-1$ at times $\tau = (k + (1/2))\Delta\tau$ are found analogously. The resulting temperature field will serve as the basis on which the temperature on the next fraction step k + 1 will be then determined. For this one uses the arithmetic circuit shown in Fig. 2b, which is now moved along the nodal lines parallel to the Y axis in the network field.

When at some nodal point (n, p) a temperature is reached equal to the solidification temperature Θ_m, one connects to the node (m, p), (n, p) of the arithmetic circuit (Fig. 2) an additional resistor R_{add} through which current $I = (1-V_m)/R_{add}$ is fed to this node so as to make the potential at node (m, p), (n, p) equal to Φ_m. The current I is supplied at each fraction step until the equality

$$\sum_{j=1}^{d}\left(\frac{1}{R_{add}}\right)_j = \frac{L_m}{1-V_m}\cdot\frac{\Delta X^2}{\Delta\tau}(\Delta Y^2) \tag{24}$$

will hold true.

Condition (24) signifies that the phase-transformation boundary is moving to the next nodal point in the network field.

Experience has shown that the proposed method is effective in solving two- and three-dimensional problems of heat conduction with moving boundaries, that it significantly reduces the number of components in the simulating device (by a factor of $1.5\sqrt{N}$, approximately, with N denoting the number of nodal points in the network field) which would be required in a resistance network according to the Liebmann method. The application of static electrical integrators, which cannot match digital computers in terms of speed and precision, may prove useful for various kinds of engineering calculations when high speed and high precision are not the main considerations.

NOTATION

$X = x/l_1$ is the dimensionless space coordinate;
$Y = y/l_1$ is the dimensionless space coordinate;
ΔX, ΔY are the increments along the respective space coordinates;
$\Delta\tau$ is the increment of time;
$N_1 = l_1/\Delta x$;
$N_2 = l_2/\Delta y$;
$\xi(\tau) = \delta(t)/l_1$ is the location of the moving boundary;
$\bar{\lambda}(\Theta) = \lambda(U)/\lambda_{max}$ is the thermal conductivity;
$c(\Theta) = c(U)/c_{min}$ is the specific heat;
$a(\Theta) = \lambda(\Theta)/c(\Theta)$
$\Theta = U/U_{max}$ is the dimensionless temperature;
T is the temperature, °K;
$\Theta_m = U_m/U_{max}$ is the phase-transformation temperatures;
$L_m = L/c_{min}U_{max}$ is the dimensionless heat of phase transformation;
$\tau = \lambda_{max}t/\rho c_{min}l_1^2$ is the dimensionless time;
$Bi = \alpha l_1/\lambda_{max}$ is the dimensionless coefficient of heat transfer;
$Bi^*(\Theta) = Bi + \varepsilon\sigma/\lambda_{max}\cdot 100[(T_{sur}/100)^3 + (T_{sur}/100)^2(T_{amb}/100) + (T_{sur}/100)(T_{amb}/100)^2 + (T_{amb}/100)^3]$;
σ is the Stefan–Boltzmann constant;
ε is the emissivity;
ρ is the density;
Φ is the defined by Eq. (7);

$$\Phi_m = \int_0^{\Theta_m}\bar{\lambda}(\Theta)\,d\Theta;$$

$\Theta_{eff} = 2\Phi_{sur} - \Theta_{sur}$;

R_0, n, R_{add}, R_a are the resistors in the arithmetic circuit, referred to the scale value R*;

R1 is the linear potentiometer;

R2 is the function potentiometer;

NG is the null-compensation galvanometer;

SP is the search plug;

$V = U/U_0$ is the voltage referred to the voltage of the potentiometer supply U_0;

d is the number of time steps through which the moving boundary is displaced by one step on the $\Delta X(\Delta Y)$ grid;

$l_{21} = l_2/l_1$

Subscripts:

$n = X/\Delta X$: 0, 1, 2, . . . , N_1;

$p = Y/\Delta Y$: 0, 1, 2, . . . , N_2;

sur refers to surface;

amb refers to surrounding medium;

m refers to phase transformation (temperature);

Superscripts:

$k = \tau/\Delta\tau$: 0, 1, 2, . . .;

s refers to the number of iteration.

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